# A CLASS OF SHEAVES SATISFYING KODAIRA'S VANISHING THEOREM

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This paper contains yet another refinement of Kodaira's vanishing theorem; the result in question (corollary 3.3) can be viewed as a Kawamata-Viehweg-Kollár type theorem for vector bundles. In order to formulate and prove this cleanly, we found it convenient to introduce the class of *geometrically acyclic* sheaves. It is the construction and study of this class (and the related class of *geometrically positive* vector bundles) that is the real subject here. The name is meant to convey two things: first of all, that the sheaves in this class are acyclic in the sense that all their higher cohomology groups vanish. Secondly, this class results from closing up the subclass of sheaves of "adjoint type" under some simple algebraic and geometric operations. These closure properties come into play when checking that a particular sheaf is geometrically acyclic. It is our contention that many known vanishing theorems can (and essentially have) been proved by doing exactly this, and we hope that geometric acyclicity provides a useful tool in the search for new ones.

We will work throughout over the field of complex numbers $\mathbb{C}$. The basic constructions and properties of geometrically acyclic sheaves are worked out in the first section. The next section introduces the notion of a geometrically positive vector bundle; these are bundles for which the naive generalization Kodaira's vanishing theorem holds. The basic result is that geometrically positive vector bundles can be obtained by taking sufficiently high powers (as described in the appendix) of vector bundles which are positive in a numerical sense. The remaining two sections contain various applications, or at any rate amplifications, of these ideas in algebra and geometry.

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## 1. Geometric Acyclicity

**Theorem 1.** *For each complex projective variety $X$, there exists a class of coherent sheaves $\mathcal{GA}(X)$ on $X$, called geometrically acyclic sheaves, satisfying:*

GA1. *If $Y$ is smooth and projective and $f : Y \to X$ is an arbitrary morphism then $(f_*\omega_Y) \otimes L \in \mathcal{GA}(X)$ for any ample line bundle $L$ on $X$.*

GA2. *If*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

*is an exact sequence of coherent sheaves on a projective variety $X$ such that $\mathcal{F}_1, \mathcal{F}_3 \in \mathcal{GA}(X)$, then $\mathcal{F}_2 \in \mathcal{GA}(X)$.*

GA3. *For each $X$, a direct summand of a sheaf in $\mathcal{GA}(X)$ is also in $\mathcal{GA}(X)$.*

GA4. *If $f : Y \to X$ is an arbitrary morphism of projective varieties, then $f_*\mathcal{F} \in \mathcal{GA}(X)$ whenever $\mathcal{F} \in \mathcal{GA}(Y)$.*

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GA5. *If $\mathcal{F} \in \mathcal{GA}(X)$, then $R^i g_* \mathcal{F} = 0$ whenever $i > 0$ and $g : X \to Z$ is a morphism of projective varieties; in particular, $\mathcal{F}$ is acyclic.*

*Proof.* Define $\mathcal{GA}_0(X)$ to be the class of sheaves of the type described in statement GA1. Inductively define $\mathcal{GA}_{n+1}(X)$ to be the class of sheaves $\mathcal{F}$ such that either $\mathcal{F}$ is an extension of sheaves in $\mathcal{GA}_n(X)$, $\mathcal{F}$ is a direct summand of a sheaf in $\mathcal{GA}_n(X)$, or $\mathcal{F} = f_*\mathcal{F}'$ where $f : Y \to X$ is a morphism of projective varieties and $\mathcal{F}' \in \mathcal{GA}_n(Y)$. Set $\mathcal{GA}(X) = \bigcup_n \mathcal{GA}_n(X)$. The first four statements clearly hold.

Statement GA5 will be proved by induction. By definition, any sheaf in $\mathcal{GA}_0(X)$ is of the form $\mathcal{F} = (f_*\omega_Y) \otimes L$ with $Y$ smooth, $f : Y \to X$ a morphism, and $L$ an ample line bundle on $X$. Given a morphism $g : X \to Z$ with $Z$ projective, choose an ample line bundle $M$ on $Z$. Then

$$H^i(X, f_*\omega_Y \otimes L \otimes g^*M^{\otimes N}) = 0$$

for $i > 0$ and $N \geq 0$ by Kollár's vanishing theorem [Ko1]. Therefore by Serre's vanishing theorem, the Leray spectral sequence and the projection formula, we obtain

$$H^0(Z, R^i g_*(f_*\omega_Y \otimes L) \otimes M^{\otimes N}) = 0$$

when $i > 0$ and $N >> 0$ . As the sheaves above can be assumed to be globally generated (again by Serre), we obtain $R^i g_*(f_*\omega_Y \otimes L) = 0$ for $i > 0$.

Now suppose that GA5 has been established for sheaves in $\mathcal{GA}_n(X)$ (for all $X$, $g$ and $i$). Choose a sheaf $\mathcal{F} \in \mathcal{GA}_{n+1}(X)$ and a morphism $g : X \to Z$. If $\mathcal{F}$ is a summand or an extension of sheaves in in $\mathcal{GA}_n(X)$, then clearly $R^i g_*\mathcal{F} = 0$ for $i > 0$. Therefore we may assume that $\mathcal{F} = f_*\mathcal{F}'$ with $\mathcal{F}' \in \mathcal{GA}_n(Y)$ with $f : Y \to X$ a morphism. By the induction hypothesis, $R^i f_*\mathcal{F}' = 0$ for $i > 0$. Therefore the spectral sequence for the composite collapses to yield isomorphisms

$$R^i g_* f_* \mathcal{F}' \cong R^i (g \circ f)_* \mathcal{F}'$$

But the sheaves on the right also vanish for $i > 0$ by induction.

□

As should be clear from the proof, $\mathcal{GA} = \bigcup_X \mathcal{GA}(X)$ is taken to be the smallest class for which the theorem holds. We expect that there are in fact natural extensions of this class for which properties GA1 to GA5 continue to hold. However, this will not be pursued here.

Let us discuss a few basic examples of geometrically acyclic sheaves.

**Example 1.1.** *Let $X$ be a smooth projective variety, and let $D \subset X$ be a reduced divisor with normal crossings. Then $\omega_X(D) \otimes L$ is geometrically acyclic, provided that $L$ is an ample line bundle. To see this, write $D = D_1 + D'$ where $D_1$ is a component. Then, using the Poincaré residue, $\omega_X(D) \otimes L$ can be expressed as an extension of $\omega_{D_1}(D') \otimes L$ by $\omega_X(D') \otimes L$. Hence the result follows by induction on the number of components.*

**Example 1.2.** *Let $X$ be a smooth projective variety. Let $D$ a $\mathbb{Q}$-divisor with normal crossing support, i.e. a sum $\sum_i a_i D_i$ with $a_i \in \mathbb{Q}$ such that $\sum_i D_i$ is a normal crossing divisor. Let $L$ be a line bundle such that $L(D)$ is nef (i.e., $c_1(L(D)) := c_1(L) + \sum_i a_i[D_i]$ has a nonnegative intersection number with any curve on $X$) and big (i.e., $c_1(L(D))^{dim X} > 0$). Then $\omega_X \otimes L(\lceil D \rceil)$ is geometrically acyclic, where the symbol $\lceil D \rceil$ means that the coefficients should be rounded up to*

*the nearest integers. The proof of this is implicit in* [Ka2]*. By using embedded resolution of singularities and a series of covering tricks, he showed that there exists a generically finite map of smooth projective varieties $\pi : X' \to X$, an ample line bundle $L'$, and a reduced normal crossing divisor $D'$ on $X'$, such that $\omega_X \otimes L(\lceil D \rceil)$ is a direct summand of $\pi_*(\omega_{X'} \otimes L'(D'))$.*

It will be convenient to extend these definitions to singular varieties. The extensions will be based on properties which are known to equivalent in the smooth case (see [Ka2, Mo]). A line bundle $L$ on a projective variety $X$ is nef provided that for any ample line bundle $H$ and positive integer $n$, $L^{\otimes n} \otimes H$ is ample. $L$ is big provided that for any ample line bundle $H$, $L^{\otimes n} \otimes H^{-1}$ has a nonzero global section for some $n > 0$. Alternatively, $L$ is big if and if $h^0(L^{\otimes n}) = O(n^{dim X})$ (after replacing $L$ by a sufficiently high power). In order to check that $L$ is big it suffices to show that $L^{\otimes n} \otimes H^{-1}$ has a nonzero global section for some $n > 0$ and a line bundle $H$ which is already known to be big.

**Lemma 1.3.** *If $\pi : X' \to X$ is a generically finite map of smooth projective varieties then $\omega_X$ is a direct summand of $\pi_* \omega_{X'}$*

*Proof.* The result is certainly well known, so we will be quite brief. There are two maps: the Grothendieck trace $\tau : \pi_*\omega_{X'} \to \omega_X$ and a map $p$ in the opposite direction which corresponds to pullback of forms. The lemma follows from the identity $\tau \circ p = deg \pi$. ☐

The third example, which will be needed later, is a slight variation on the previous ones.

**Lemma 1.4.** *Let $f : Y \to X$ be a surjective morphism of smooth projective varieties, and $L$ a nef and big line bundle on $X$, then $f_*\omega_Y \otimes L$ is geometrically acyclic.*

*Proof.* In the course of the proof, we will construct a large commutative diagram:

$$\begin{array}{ccccc} \cdots & Y_i & \to & \cdots & Y \\ \cdots & f_i \downarrow & & \cdots & \downarrow \\ \cdots & X_i & \to & \cdots & X \end{array}$$

We will denote the maps $X_i \to X_{i-1}$, $X_i \to X$, $Y_i \to Y_{i-1}$ and $Y_i \to Y$ by $p_i$, $P_i$, $\pi_i$ and $\Pi_i$ respectively.

Let $H$ be a very ample divisor on $X$. If $n >> 0$, $L^{\otimes n}(-H)$ has a nonvanishing section, and let $D$ be the corresponding effective divisor. Let $X_1 \to X$ be a birational map of smooth varieties such that the pullback $D_1$ of $D$ has normal crossings. Let $Y_1$ be a desingularization of $Y \times_X X_1$ such that $f^{-1}D_1$ also has normal crossings [Hi].

We can choose a positive rational linear combination of exceptional divisors $E_1$ such that $H_1 = P_1^{-1}H - E_1$ is an ample $\mathbb{Q}$-divisor. Define

$$\begin{aligned} H_1' &= \epsilon H_1 + (1-\epsilon)P_1^{-1}(H + D) \\ &= H_1 + (1-\epsilon)(E_1 + D_1) \end{aligned}$$

where $0 < \epsilon << 1$ is rational, then $H_1'$ is an ample $\mathbb{Q}$-divisor. Therefore

$$P_1^{-1}(H + D) = H_1' + \epsilon(E_1 + D_1).$$

Therefore, after replacing $n$ by a large multiple, we can assume that the pullback of the linear system associated to $L^{\otimes n}$ has an integral normal crossing divisor

of the form $H_1'' + D_1''$ where $H_1''$ is smooth, ample and transverse to $Y_1 \to X_1$, and multiplicities of the components of $D_1''$ and $f^{-1}D_1''$ are less than $n$. Apply Kawamata's trick [Ka1, thm 17], to obtain a finite cover $X_2 \to X_1$, branched over $H_1''$ such that the pullback $H_2$ of $H_1''$ has multiplicity $n$ and the multiplicities of the components of the pullback $D_2$ of $D_1''$ are unchanged. Let $f_2 : Y_2 \to X_2$ be the fiber product of $Y_1$ with $X_2$. By our assumptions, $Y_2$ is smooth and $f_2^{-1}(H_2 + D_2)$ is a normal crossing divisor such that $f_2^{-1}H_2$ has multiplicity $n$ and all other components have smaller multiplicity.

Let $Y_3 \to Y_2$ be a desingularization of the $n$-fold cyclic cover branched along $f_2^{-1}(H_2 + D_2)$ determined by the line bundle $f_2^*P_2^*L$ (see [EV, V]). Using the formula for the canonical sheaf of a cyclic cover [loc. cit.], we see that

$$\pi_{3*}\omega_{Y_3} = \bigoplus_{i=0}^{n-1} \omega_{Y_2} \otimes f_2^*P_2^*L^{\otimes i}(-[\frac{i}{n}f_2^{-1}(H_2 + D_2)]).$$

By our assumptions about multiplicity, the summand on the right for $i = 1$ is just $\omega_{Y_2} \otimes f_2^*P_2^*L(-f_2^{-1}H_2)$. By lemma 1.3, we see that $f_*\omega_Y \otimes L$ is a direct summand of $f_*\Pi_{2*}(\omega_{Y_2} \otimes f_2^*P_2^*L)$ which is in turn a direct summand of $P_{2*}((f_2 \circ \pi_3)_*\omega_{Y_3} \otimes O(H_2))$. Therefore it is geometrically acyclic.

□

## 2. Geometric Positivity

Let us say that a vector bundle $E$ is *geometrically semipositive* provided that $\mathcal{F} \otimes E$ is geometrically acyclic whenever $\mathcal{F}$ is. We will say that a vector bundle $E$ on $X$ is *geometrically positive* provided that it is geometrically semipositive, and $f_*\omega_Y \otimes E$ is geometrically acyclic whenever $f : Y \to X$ is a surjective map from a smooth projective variety.

**Example 2.1.** *A nef line bundle $L$ is geometrically semipositive. This follows by induction: If $\mathcal{F} \in \mathcal{GA}_0(X)$, then $\mathcal{F} \otimes L \in \mathcal{GA}_0(X) \subset \mathcal{GA}(X)$ because $L \otimes H$ is ample whenever $H$ is an ample line bundle. Suppose that the result is known for all $\mathcal{F} \in \mathcal{GA}_n = \bigcup_Y \mathcal{GA}_n(Y)$, then clearly it holds for a direct summand or an extension of a sheaves in $\mathcal{GA}_n$. If $\mathcal{F} = f_*\mathcal{F}'$ with $\mathcal{F}' \in \mathcal{GA}_n$, then $\mathcal{F} \otimes L = f_*(\mathcal{F}' \otimes f^*L) \in \mathcal{GA}$.*

*This together with lemma 1.4 shows that a nef and big line bundle is geometrically positive.*

It should be clear that the class of geometrically (semi-)positive vector bundles is stable under extensions, direct summands, and tensor products, and therefore also under the Schur-Weyl powers (see appendix).

**Lemma 2.2.** *If $E$ is geometrically semipositive (respectively positive), then so is $\mathbb{S}^\lambda(E)$ for any (nonzero) partition $\lambda$.*

*Proof.* $\mathbb{S}^\lambda(E)$ is a direct summand of a tensor power of $E$. □

Let us say that a vector bundle $E$ on a projective variety $X$ is nef provided that $O_{\mathbb{P}(E)}(1)$ is nef. (We use the convention that $\mathbb{P}(E) = \mathbf{Proj}(S^\bullet(E))$.) It is easy to see that locally free quotients of nef vector bundles, in particular globally generated vector bundles, are nef with this definition. Less obvious is the fact that direct sums and tensor products of nef bundles are nef. This can deduced easily from the following lemma and the ampleness of direct sumas, tensor products and quotients of ample vector bundles [Ha].

**Lemma 2.3.** *Let $H$ be an ample line bundle on $X$. $E$ is nef if and only if $S^n(E)\otimes H$ is ample for all $n >> 0$.*

*Proof.* Suppose that $E$ satisfies the condition of the lemma, but that $O_{\mathbb{P}(E)}(1)$ fails to be nef. Then there exists a curve $C \subset \mathbb{P}(E)$, with $c_1(O(1)) \cdot [C] < 0$. Since $O(n) \otimes \pi^* H$ is quotient of $\pi^*[S^n(E) \otimes H]$ where $\pi$ is the projection,

$$c_1(O(1)) \cdot [C] + \frac{1}{n}\pi^* c_1(H) \cdot [C] \geq 0.$$

This yields a contradiction when $n \to \infty$.

Conversely suppose that $E$ is nef. Nakai-Moishezon's criterion shows that $O(1)\otimes \pi^* H$ is ample, therefore $E \otimes H$ is ample. There exists a finite branched cover $p : Y \to X$ such that $p^* H$ possesses an $n$th root [BG], that is a line bundle $L$ such that $L^{\otimes n} \cong p^* H$. Therefore $p^*(S^n E \otimes H) \cong S^n(p^* E \otimes L)$ is ample, and this implies that $S^n E \otimes H$ is ample. □

The property of being nef is also known as numerical semipositivity. The two notions of semipositivity are related:

**Lemma 2.4.** *A geometrically semipositive bundle $E$ on a smooth projective variety $X$ is nef.*

*Proof.* Let $O_X(1)$ be a very ample line bundle such that $H = \omega_X \otimes O_X(dim\, X + 2)$ is also ample. Then $H^i(X, S^n(E) \otimes H(-1-i)) = 0$ for all $i > 0$ and $n > 0$. In other words, $S^n(E) \otimes H$ is $(-1)$-regular [Mu, p 100]. This implies that $S^n(E) \otimes H$ is a quotient of a sum of $O_X(1)$'s, and therefore ample. Thus $E$ is nef by the previous lemma.

□

We call a vector bundle $E$ big provided that there exists a very ample line bundle $H$ and $n > 0$, such that $S^n(E) \otimes H^{-1}$ is generically generated by global sections, which means that

$$H^0(S^n(E) \otimes H^{-1}) \otimes O_X \to S^n(E) \otimes H^{-1}$$

is surjective over a nonempty open set

When $E$ is big, $S^n(E)\otimes L^{-1}$ is generically globally generated for any line bundle $L$ and some $n > 0$, because a negative power of $H$ injects into $L^{-1}$. The pullback of an ample bundle under a birational map is both nef and big. This notion of bigness coincides with the previous one for line bundles, and agrees with a more general one for torsion free sheaves [Mo, pp 292-293]. Bigness of $E$ is a stronger condition than the bigness of $O_{\mathbb{P}(E)}(1)$ as the following shows:

**Lemma 2.5.** *Suppose that $E$ is a big vector bundle over a projective variety $X$. Let $Y$ be projective variety with a map $f : Y \to \mathbb{P}(E)$ which is generically finite over $f(Y)$ and such that the composite $Y \to X$ is surjective, then $f^* O_{\mathbb{P}(E)}(1)$ is big.*

*Proof.* Denote the projection $\mathbb{P}(E) \to X$ by $\pi$. Let $H$ be an ample line bundle on $\mathbb{P}(E)$ which is necessarily of the form $O_{\mathbb{P}(E)}(a) \otimes \pi^* L$ where $L$ is a line bundle on $X$. As $f^* H$ is big, it is enough to check that $f^*(O_{\mathbb{P}(E)}(N) \otimes H^{-1})$ has a nonzero section for some $N > 0$ (see the comments preceding lemma 1.3).

Let $y$ be a general point of $Y$; it lies over a general point $x \in X$. $\mathbb{C}(x)$ will denote the residue field at $x$. Choose a trivialization of $L$ at $x$. Then a section

of $S^nE_x \otimes L_x^{-1} \otimes \mathbb{C}(x)$ can be identified with a section of $S^nE_x \otimes \mathbb{C}(x)$, and this determines a hypersurface in the fiber $\mathbb{P}(E)_x$. Choose $n > 0$ so that all elements of $S^nE_x \otimes L_x^{-1}$ extend to global sections, and choose a section $\sigma' \in S^nE_x \otimes L_x^{-1} \otimes \mathbb{C}(x)$, so that the corresponding hypersurface avoids $f(y)$. Then $\sigma'$ lifts to a global section of $S^nE \otimes L^{-1}$ which can be identified with a global section $\sigma$ of $O_{\mathbb{P}(E)}(n+a) \otimes H^{-1}$. Pulling $\sigma$ back to $Y$ yields a nonvanishing section of $f^*(O_{\mathbb{P}(E)}(n+a) \otimes H^{-1})$. □

**Lemma 2.6.** *A locally free quotient of a big vector bundle is big.*

*Proof.* Let $F$ be a locally free quotient of a big vector bundle $E$. Suppose $H$ is very ample, choose $n > 0$ so that $S^nE \otimes H^{-1}$ is generically generated by its global sections. Then the restriction of these sections also generates $S^nF \otimes H^{-1}$ generically. □

**Lemma 2.7.** *If $E$ is big, then for any coherent sheaf $\mathcal{F}$ there exists an integer $n_0$ such that $S^n(E) \otimes \mathcal{F}$ is generically globally generated whenever $n \geq n_0$.*

*Proof.* The proof will be reduced to a series of observations. Fix a very ample line bundle $H$.

The tensor product of two or more generically globally generated coherent sheaves has the same property. Therefore the set $I(H)$ of integers $n$ for which $S^n(E) \otimes H^{-1}$ is generically globally generated forms a semigroup.

Given a coherent sheaf $\mathcal{F}$, choose an integer such that $\mathcal{F} \otimes H^{\otimes m}$ is globally generated. Then $S^n(E) \otimes \mathcal{F}$ is generically globally generated for any $n \in I(H^{\otimes m}) \subset I(H)$.

Apply the result of the previous paragraph to obtain $n \in I(H)$ such that $S^n(E) \otimes E \otimes H^{-1}$ is generically globally generated. This implies that $n+1 \in I(H)$. A semigroup containing two relatively prime integers contains all but finitely many positive integers, and this concludes the proof.

□

**Lemma 2.8.** *The direct sum of two big vector bundles is big.*

*Proof.* Let $E_1$ and $E_2$ be two big vector bundles and $H$ a very ample line bundle. Then for each $m \geq 0$, there is an integer $N_m > 0$ such that for both $i$ and all $n \geq N_m$, $S^n(E_i) \otimes H^{-\otimes m}$ is generically globally generated. Choose $M > 0$ so that $S^n(E_i) \otimes H^{\otimes M}$ is globally generated for all $n < N_1$. Then choose $R \geq 2N_{M+1}$. Then one sees, after grouping terms appropriately, that

$$S^n(E_1) \otimes S^{R-n}(E_2) \otimes H^{\otimes -(1+M)} \otimes H^{\otimes M}$$

is generically globally generated for any $n \geq 0$. Therefore the same holds for $S^R(E_1 \oplus E_2) \otimes H^{-1}$.

□

**Lemma 2.9.** *A Schur-Weyl power of a big vector bundle is big. The tensor product of two big vector bundles is big.*

*Proof.* Suppose that $E$ is a big vector bundle, and $H$ an ample line bundle. For any partition $\lambda$, there exists integers $N_1, \dots N_r$ such that $min(N_i) \to \infty$ as the weight $|\lambda| \to \infty$ and such that $\mathbb{S}^\lambda(E)$ is a direct summand of

$$S^{N_1}(E) \otimes \dots \otimes S^{N_r}(E)$$

by [Ha, 5.1]. This implies that $\mathbb{S}^\lambda(E)\otimes H^{-1}$ is generically globally generated for $|\lambda| >> 0$. Therefore $S^N(\mathbb{S}^\lambda(E))\otimes H^{-1}$ is generically globally generated for arbitrary $\lambda$ and $N >> 0$, since the first factor can be decomposed into a sum of Schur-Weyl powers of large weight.

If $E$ and $F$ are both big, then $E\otimes F$ must be big since it is a direct summand of $S^2(E\oplus F)$.

□

The converse to lemma 2.4 is false as we will see (example 3.9). However, the following may be viewed as a weak converse. (The notation is explained in the appendix.)

**Theorem 2.** *Let $E$ be a vector bundle on a projective variety and $\lambda$ a nonzero partition.*

1. *If $E$ is nef, then $\mathbb{S}^\lambda_+(E)$ is geometrically semipositive.*
2. *If $E$ is nef and big, then $\mathbb{S}^\lambda_+(E)$ is geometrically positive.*

Before giving the proof, we will need the following lemmas.

**Lemma 2.10.** *Suppose that $k_1 < \ldots < k_m$ is a sequence of positive integers. Let $E$ be a rank $k_{m+1}$ vector bundle over a variety $X$, let $p : F = Flag_{k_1,k_2\ldots k_m}(E)\to X$ be the bundle of partial flags, and let $\pi_i : F\to Grass_{k_i}(E)$ be the natural projections. If we set $k_0 = 0$, then*

$$\omega_F = p^*\omega_X\otimes p^*(detE)^{\otimes k_m}\otimes\pi_1^*O_{Grass_{k_1}}(k_0-k_2)\otimes\ldots\pi_m^*O_{Grass_{k_m}}(k_{m-1}-k_{m+1})$$

*where $O_{Grass_{k_i}}(1)$ are the restrictions of the hyperplane bundles under the Plücker embeddings.*

*Proof.* See [D, 2.10] □

**Lemma 2.11.** *Let $E, F, X$ be as in the previous lemma, and $a_1,\ldots a_m$ be positive integers. Then $\pi_1^*O_{Grass_{k_1}}(a_1)\otimes\ldots\pi_m^*O_{Grass_{k_m}}(a_m)$ is nef (respectively nef and big) if $E$ is nef (respectively nef and big).*

*Proof.* Let $P = \mathbb{P}(\wedge^{k_1}E)\times_X\ldots\times_X\mathbb{P}(\wedge^{k_m}E)$ and let $i : F\hookrightarrow P$ denote the Plücker embedding. The restriction of $M(a_1,\ldots,a_m) = O(a_1)\otimes\ldots O(a_m)$ to $F$ is $\pi_1^*O_{Grass_{k_1}}(a_1)\otimes\ldots\pi_m^*O_{Grass_{k_m}}(a_m)$. If $E$ is nef, then so is $M(a_1,\ldots,a_m)$, and hence also $i^*M(a_1,\ldots a_m)$. This argument requires only that all $a_j\geq 0$.

Suppose that $E$ is nef and big, then $E' = \wedge^{k_1}E\oplus\ldots\oplus\wedge^{k_m}E$ is also nef and big by the preceding lemmas. Let $j : P\hookrightarrow\mathbb{P}(E')$ be the Segre embedding. Then $i^*j^*O_{\mathbb{P}(E')}(1) = i^*M(1,\ldots 1)$ is nef and big by lemma 2.5. This together with the previous paragraph implies that

$$i^*M(a_1,\ldots a_m) = i^*M(1,\ldots 1)\otimes i^*M(a_1-1,\ldots a_m-1)$$

is again nef and big. □

*Proof of theorem.* Let $k_1 < \ldots < k_m$ be the list of indices $k$ for which $\lambda_k-\lambda_{k+1}\neq 0$, and let $a_i = \lambda_{k_i}-\lambda_{k_i+1}$. Also let $k_{m+1} = rk(E)$ and $k_0 = 0$.

Let $F = Flag_{k_1,\ldots k_m}(E)$ and let

$$O_F(b_1, b_2,\ldots) = \pi_1^*O_{Grass_{k_1}}(b_1)\otimes\pi_2^*O_{Grass_{k_2}}(b_2)\otimes\ldots$$

Then $M = O_F(a_1+k_2-k_0, a_2+k_3-k_1,\ldots)$ is nef or nef and big according to whether $E$ is.

Let us suppose that $E$ is nef and prove that $\mathcal{F}\otimes\mathbb{S}^\lambda_+(E)\in\mathcal{GA}(X)$ whenever $\mathcal{F}\in\mathcal{GA}_n(X)$ by induction. Suppose that $\mathcal{F}=f_*\omega_Y\otimes L$ where $Y$ is smooth and $L$ is an ample line bundle on $X$. There is no loss of generality in assuming that $f$ is surjective, since otherwise we can replace $X$ by its image. Consider the Cartesian diagram:

$$\begin{array}{ccc} F' & \xrightarrow{f'} & F \\ p'\downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

where $F'=Flag_{k_1,\ldots k_m}(f^*E)$. By lemma 2.10,

$$\omega_{F'}\otimes {f'}^*M = {p'}^*(\omega_Y\otimes det(E)^{\otimes k_m})\otimes O_{F'}(a_1,a_2,\ldots).$$

Therefore

$$p'_*(\omega_{F'}\otimes {f'}^*M)=\omega_Y\otimes\mathbb{S}^\lambda_+(f^*E)$$

$M\otimes p^*L$ is nef and also big because the restriction of $M$ to any fiber is ample so that

$$c_1(M\otimes p^*L)^{dimF}\geq c_1(M)^{dimF-dimX}p^*c_1(L)^{dimX}>0.$$

Therefore

$$f_*\omega_Y\otimes L\otimes\mathbb{S}^\lambda_+(E)\cong p_*(f'_*\omega_{F'}\otimes M\otimes p^*L).$$

is geometrically acyclic by lemma 1.4.

Suppose that we have proved the result for all sheaves in $\mathcal{GA}_n$. Then clearly it holds for a direct summand or an extension of sheaves in $\mathcal{GA}_n$. If $\mathcal{F}=f_*\mathcal{F}'$ with $\mathcal{F}'\in\mathcal{GA}_n$, then $\mathcal{F}\otimes\mathbb{S}^\lambda_+(E)=f_*(\mathcal{F}'\otimes f^*\mathbb{S}^\lambda_+(E))\in\mathcal{GA}$.

Now suppose that $E$ is both nef and big. Let $f:Y\to X$ be a surjective morphism from a smooth projective variety $Y$. Construct a Cartesian square as above. By lemma 2.11 $M$ is nef and big, therefore

$$f_*\omega_Y\otimes\mathbb{S}^\lambda_+(E)\cong p_*(f'_*\omega_{F'}\otimes M)$$

is geometrically acyclic. □

**Remark 2.12.** *It is possible to improve the theorem slightly by replacing the bigness condition on $E$ with the weaker condition that $M$ above is big. This last condition is expressible as the positivity of an appropriate Chern polynomial of $E$. The necessary condition, along with a related vanishing theorem, can be found in* [M1].

By a similar argument, we have:

**Example 2.13.** *Let $f:Y\to X$ be a smooth projective morphism then $E=f_*\omega_{Y/X}$ is a geometrically semipositive vector bundle.*

## 3. Corollaries of Theorem 2

We have finished the hard work; now for the entertainment. The first corollary is really just a reformulation of the theorem. (See appendix for the definition of $Pos(e)$.)

**Corollary 3.1.** *Let $E$ be a rank $e$ vector bundle, and $\lambda\in Pos(e)$. Then $\mathbb{S}^\lambda(E)$ is geometrically semipositive if $E$ is nef, and $\mathbb{S}^\lambda(E)$ is geometrically positive if $E$ is nef and big.*

*Proof.* Let $\lambda' = (\lambda_1 - \lambda_e, \dots)$ and $n = \lambda_e - length(\lambda')$, $n$ is positive by the definition of $Pos(e)$. Then $\mathbb{S}^\lambda(E) = \mathbb{S}^\lambda_+(E) \otimes (detE)^{\otimes n}$. As $det(E)$ is nef, the corollary is immediate. □

**Lemma 3.2.** *Let $E$ be a geometrically positive vector bundle on a projective variety $X$. If $f : Y \to X$ is a morphism from a projective variety with at worst rational singularities, then $R^i f_*\omega_Y \otimes E$ is geometrically acyclic for each $i$.*

*Proof.* By the assumption on the singularities, we can replace $Y$ by a desingularization without affecting $R^i f_*\omega_Y$. Then by [Ko2, 2.24], there exists a morphism from a smooth projective variety $g : Z \to X$ such that $R^i f_*\omega_Y$ is a direct summand of $g_*\omega_Z$. □

The next result follows directly from the previous corollary and lemma. When the $E_i$ have rank one, this is essentially the Kollár vanishing theorem. For higher rank, this has some overlap with a theorem of Manivel [M4] (which supersedes the vanishing theorems of [G], [D] and others - see the introduction to his paper); the overlap occurs when $Y = X$ is smooth and the $E_i$ are ample.

**Corollary 3.3.** *If $f : Y \to X$ is a morphism of projective varieties such that $Y$ has at worst rational singularities and if $E_i$ is a collection of nef vector bundles and $\lambda(i) \in Pos(rk(E_i))$ a collection of partitions, with $i = 1, 2 \dots n$, then*

$$H^i(X, R^j f_*\omega_Y \otimes \mathbb{S}^{\lambda(1)}(E_1) \otimes \dots \mathbb{S}^{\lambda(n)}(E_n)) = 0$$

*for all $i > 0$ and all $j$, provided that at least one of the $E_i$ is also big.*

It is possible to define an intermediate notion between geometric positivity and semipositivity similar in spirit to Sommese's $k$-ampleness [S]. Although we will not develop this systematically here, we point out a special case:

**Lemma 3.4.** *Let $f : Y \to X$ be a surjective map of projective varieties with $Y$ smooth. Let $E$ be a geometrically positive vector bundle on $X$, then*

$$H^i(Y, \omega_Y \otimes f^*E) = 0$$

*for $i > dim\, Y - dim\, X$.*

*Proof.* From the Leray spectral sequence, it suffices to kill $H^p(R^q f_*\omega_Y \otimes E)$ for all $p + q > k = dim\, X - dim\, Y$. In fact, these groups vanish for all $p > 0$ by the previous lemma. And they vanish for $q > k$, by [Ko1, 2.1] □

**Corollary 3.5.** *Let $f : Y \to X$ be a surjective map of projective varieties with $Y$ smooth. Let $E$ be a nef and big vector bundle on $X$, then*

$$H^i(Y, \omega_Y \otimes \mathbb{S}^\lambda(f^*E)) = 0$$

*for $i > dim\, Y - dim\, X$ and $\lambda \in Pos(rk(E))$.*

Recall that a global section of a vector bundle is *regular*, if its components with respect to any local basis forms a regular sequence. The defining ideal of the scheme of zeros of this section is locally generated by these components.

**Lemma 3.6.** *Let $X$ be a projective variety. Let $Z$ be the scheme of zeros of a regular section of a geometrically positive vector bundle $E$ on $X$. Let $F^*$ be geometrically semipositive vector bundle. The restriction maps*

$$H^i(X, F) \to H^i(Z, F \otimes O_Z)$$

*are isomorphisms for $i < dim\, X - rk(E)$ and and injection for $i = dim\, X - rk(E)$.*

*Proof.* Since the section is regular, we have the Koszul resolution:

$$\ldots \wedge^2 E^* \to E^* \to O_X \to O_Z \to 0.$$

This can be broken up into a series of short exact sequences. $\wedge^k E \otimes F$ is geometrically positive because it is a summand of $E^{\otimes k} \otimes F$. Therefore the lemma follows by tensoring these sequences by $F^*$, and using the vanishing of

$$H^i(X, \wedge^k E^* \otimes F^*) \cong H^{dim X - i}(X, \omega_X \otimes \wedge^k E \otimes F)^*$$

for $i < dim\, X$ □

**Corollary 3.7.** *With the same notation as in the lemma, if $X$ is irreducible then $Z$ is connected provided that $rk(E) < dim\, X$.*

*Proof.* $H^0(O_X) \cong H^0(O_Z)$. □

**Corollary 3.8.** *Let $E$ be a globally generated vector bundle on a smooth projective variety $X$. Then there is an integer $0 \le \kappa \le dim\, X$ such that for any partition $\lambda$, the multiplication map*

$$H^0(X, det(E)^{\otimes m}) \otimes H^i(X, \omega_X \otimes \mathbb{S}^\lambda(E) \otimes det(E)^{\otimes n}) \to H^i(X, \omega_X \otimes \mathbb{S}^\lambda(E) \otimes det(E)^{\otimes m+n})$$

*is surjective for all $i$, $m \ge 0$, and $n \ge length(\lambda) + \kappa$.*

*Proof.* Let $f : X \to G = Grass_r(H^0(E))$ be the map such that $E$ is the pullback of the universal quotient bundle $Q$. Set $\kappa = dim(f(X))$, and let $i : G \hookrightarrow \mathbb{P}$ denote the Plücker embedding. By the corollary 3.3 the Leray spectral sequence collapses to yield isomorphisms

$$H^i(X, \omega_X \otimes \mathbb{S}^\lambda(E) \otimes det(E)^{\otimes n}) \cong H^0(G, R^i f_* \omega_X \otimes \mathbb{S}^\lambda(Q) \otimes O_G(n)) \tag{1}$$

for $n \ge length(\lambda)$. Furthermore, the vanishing theorem implies that

$$\mathcal{F} = i_*[R^i f_* \omega_X \otimes \mathbb{S}^\lambda(Q) \otimes O_G(length(\lambda) + \kappa)]$$

is 0-regular [Mu]. Therefore by [loc. cit., page 100],

$$H^0(\mathbb{P}, O(m)) \otimes H^0(\mathbb{P}, \mathcal{F}) \to H^0(\mathbb{P}, \mathcal{F}(m))$$

is surjective for $m \ge 0$. A simple diagram chase using the maps

$$H^0(\mathbb{P}, O(m)) \to H^0(X, det(E)^{\otimes m})$$

and the isomorphism (1) finishes the proof. □

Refinements of the above idea will appear in the forthcoming work of J.Chipalkatti.

An interesting problem is to find intrinsic criteria for a vector bundle to be geometrically (semi-)positive. Simple examples show that the condition of being nef and big is not sufficient:

**Example 3.9.** *The ith exterior power of the tangent bundle $E = \wedge^i T_P$ of $P = \mathbb{P}^n$ is ample. However it cannot be geometrically positive (for $i < n$) because*

$$H^{n-i}(P, \omega_P \otimes E) = H^i(P, \Omega_P^i)^* \neq 0$$

*This also shows that $E(-1)$ is not geometrically semipositive even though it is globally generated (and even ample for $i > 1$).*

Let us say that a vector bundle $E$ of rank $r$ is *strongly semistable* if and only if $S^r(E)\otimes det(E)^{-1}$ is nef. The terminology will be explained shortly. But first let us point out that one can build simple examples using the following observations: $E$ is strongly semistable if and only if $E\otimes L$ is strongly semistable for any line bundle $L$, and a nef vector bundle $E$ is strongly semistable if $c_1(E)=0$.

**Corollary 3.10.** *Let $E$ be a strongly semistable vector bundle. Then $E$ is geometrically semipositive if $det(E)$ is nef, and $E$ is geometrically positive if $det(E)$ is nef and big.*

*Proof.* Since these conditions are stable by pullback under a generically finite map, there is no harm in assuming that the base variety $X$ is smooth and projective. Let $r=rk(E)$ then there exists a smooth variety $Y$ with a line bundle $L$ and a finite map $p:Y\to X$ such that $p^*det(E)=L^{\otimes r}$. The existence of $Y$ follows from [BG] or [Ka1, thm 17]. Therefore $S^r(F)$ is nef, where $F=p^*E\otimes L^{-1}$. The Veronese embedding $\mathbb{P}(F)\hookrightarrow \mathbb{P}(S^r(F))$ shows that $O_{\mathbb{P}(F)}(r)$ is also nef. Consequently, so is $F$. The theorem implies that $F=F\otimes det(F)=\mathbb{S}_+^{(1,0\ldots)}(F)$ is geometrically semipositive. Since $E$ is a direct summand of $p_*(F\otimes L)$, the corollary follows. ☐

As for the name:

**Lemma 3.11.** *If $E$ is a strongly semistable vector bundle on a smooth projective variety $X$, then $E$ is semistable with respect to any polarization. In other words, given an ample line bundle $H$ and a torsion free quotient $F$,*

$$\frac{c_1(F)c_1(H)^{dim X-1}}{rk(F)}\geq\frac{c_1(E)c_1(H)^{dim X-1}}{rk(E)}.$$

*Proof.* We will only sketch the proof since the lemma is not used here. Proceed as above to construct a map $p:Y\to X$ of smooth projective varieties such that $p^*det(E)$ has an $rk(E)$th root $L$. Then $p^*E\otimes L^{-1}$ is nef, hence also is its restriction to any complete intersection curve $C$ corresponding to a multiple of $p^*H$. The same goes for $(p^*F\otimes L^{-1})|_C$, so it has nonnegative degree. This last condition is equivalent to the above inequality. ☐

Strong semistablitiy is known to be equivalent to ordinary semistablity for curves [Mi, sect. 3]. The set of strongly semistable bundles in an irreducible component of the moduli space of semistable bundles (for a fixed polarization) is easily seen to be either empty or the complement of a countable union of proper subvarieties.

Manivel [M3] has introduced the class of uniformly nef vector bundles on projective varieties. It is the smallest class containing the bundles $E\otimes L$, where $E$ is unitary flat and $L$ a nef line bundle, and which is closed under extensions, direct summands, and such that $E$ is uniformly nef if and only if its inverse image under a finite map is.

**Corollary 3.12.** *A uniformly nef vector bundle is geometrically semipositive.*

*Proof.* By corollary 3.10 and the comments preceding it, a vector bundle of the form $E\otimes L$, with $E$ unitary flat and $L$ a nef line bundle, is both nef and strongly semistable and therefore geometrically semistable. Moreover a vector bundle is clearly nef and strongly semistable if and only if its pullback under a finite map is. This along with the fact that the class of geometrically semipositive vector bundles is closed under extensions and direct summands implies the corollary. ☐

Let $X$ be a smooth projective variety. It is possible to define a countable collection of invariants for $X$ as follows. Given a partition $\lambda$ of length less than or equal to $dim\, X$, define the associated Schur-Weyl invariant by

$$q_\lambda(X) = dim\, H^0(X, \mathbb{S}^\lambda \Omega^1_X).$$

These numbers, which include the plurigenera, are birational invariants. A detailed study of these invariants can be found in [M2].

**Corollary 3.13.** *Suppose that $\Omega^1_X$ is nef and that $\omega_X$ is big . If $\lambda$ is a partition such that*

$$\lambda_{dimX} \geq length(\lambda_1 - \lambda_{dimX}, \ldots \lambda_{dimX-1} - \lambda_{dimX}, 0, \ldots) + 2,$$

*then $q_\lambda(X)$ is multiplicative under étale covers and invariant under small deformations.*

*Proof.* By our assumptions, there exists a partition $\lambda' \in Pos(dim\, X)$, such that

$$\mathbb{S}^\lambda \Omega^1_X = \omega_X^{\otimes 2} \otimes \mathbb{S}^{\lambda'} \Omega^1_X.$$

Consequently, $q_\lambda(X) = \chi(\mathbb{S}^\lambda \Omega^1_X)$. The right hand side is multiplicative and a deformation invariant by Riemann-Roch. Furthermore, this equality persists under étale covers because the pullback of a nef (and big) vector bundle is nef (and big), and it persists under small deformations by upper semicontinuity of cohomology. □

Varieties satisfying the conditions of the previous corollary are easy to construct by taking subvarieties of an abelian variety of general type. If $\Omega^1_X$ is nef and big (e.g., ample) then the conclusion of the corollary holds for any partition satisfying

$$\lambda_{dimX} \geq length(\lambda_1 - \lambda_{dimX}, \ldots \lambda_{dimX-1} - \lambda_{dimX}, 0, \ldots) + 1.$$

## 4. Blow ups of coherent sheaves, and local algebra

Fix a local domain $(R, m)$, with fraction field $K$, of essentially finite type over $\mathbb{C}$. When no (or only moderate) confusion is likely, we denote an $R$-module and the associated quasicoherent sheaf on $spec\, R$ by the same symbol. Given a coherent sheaf $E$ on an integral scheme $Z$, we will say that a birational morphism $f : X \to Z$ *frees* $E$ if the quotient of $f^*E$ by its torsion submodule is locally free.

**Lemma 4.1.** *Let $Z$ be either an irreducible algebraic variety over $\mathbb{C}$ or $spec\, R$. Given a sequence $E_1, \ldots E_n$ of coherent $O_Z$-modules, there exists a resolution of singularities $f : X \to Z$ which frees all the $E_i$.*

*Proof.* Let $r_i$ be the rank of $E_i$ at the generic point. Let $f_1 : X_1 \to spec\, R$ be the blow up of the $r_1$st Fitting ideal of $E_1$. Let $f_2 : X_2 \to spec\, R$ be the blow up of the $r_2$nd Fitting ideal of $f_1^* E_2$, and so on. By [GR, p. 40], $X_n$ frees all of the $E_i$. To finish the construction, choose a desingularization $X \to X_n$. □

**Theorem 3.** *Let $f : X \to spec\, R$ be a resolution of singularities which frees a sequence $E_1, \ldots E_n$ of finitely generated $R$-modules, and let $D$ be the exceptional divisor. Let $\tilde{E}_i$ be the quotient of $f^*E_i$ by its torsion submodule. Suppose that $\lambda_i \in Pos(dim\, E_i \otimes K)$. Then*

$$H^i(X, \omega_X \otimes \mathbb{S}^{\lambda_1}(\tilde{E}_1) \otimes \ldots \mathbb{S}^{\lambda_n}(\tilde{E}_n)) = 0 \tag{1}$$

*for all $i > 0$, and*

$$H^i_D(X, \mathbb{S}^{\lambda_1}(\tilde{E}^*_1) \otimes \ldots \mathbb{S}^{\lambda_n}(\tilde{E}^*_n)) = 0 \tag{2}$$

*for all $i < dim R$.*

*Proof.* We first make a few preliminary observations. (2) follows from (1) by duality [L, p 188]. Next suppose that $\pi : X' \to X$ is a birational map with $X'$ smooth, then the equality $\mathbb{R}\pi_*\omega_{X'} = \omega_X$ along with the projection formula shows that it is enough to prove the (1) result for $X'$.

The heart of the argument involves globalizing. Let $Z$ be a projective variety with a point $z$ such that $R \cong O_{Z,z}$. Choose presentations $R^{n'_i} \to R^{n_i} \to E_i \to 0$ for each $i$. The presentation matrices can be extended to matrices of regular functions in a neighbourhood of $z$. Hence after blowing up $Z$ (away from $z$) these matrices extend to maps $A_i : O_Z^{n'_i} \to O_Z(D)^{n_i}$ where $D$ is an effective Cartier divisor containing all the poles of the matrix entries and such that $z \notin D$. We can assume that $D$ is ample, since otherwise we can replace it with $D + NH$ with $N >> 0$, where $H$ is a very ample divisor avoiding $z$. Set $\mathcal{E}_i = coker(A_i)$. Then by construction $\mathcal{E}_{iz} = E_i$. Let $\bar{f} : \bar{X} \to Z$ be a resolution of singularities which frees the sheaves $\mathcal{E}_i$. By blowing up further, we can assume that $X' = \bar{X} \times_Z spec\, R$ dominates $X$. The key point is that we arranged that each $\tilde{\mathcal{E}}_i = \bar{f}^*\mathcal{E}_i/torsion$ is a quotient of a direct sum of $f^*O_Z(D)$; as $D$ is ample, this implies that $\tilde{\mathcal{E}}_i$ is nef and big. Therefore by theorems 1 and 2,

$$R^i\bar{f}_*(\omega_{\bar{X}} \otimes \mathbb{S}^{\lambda_1}(\tilde{\mathcal{E}}_1) \otimes \ldots \mathbb{S}^{\lambda_n}(\tilde{\mathcal{E}}_n)) = 0.$$

We obtain (1) for $X'$ since cohomology commutes with flat base change.

□

**Corollary 4.2.** *Suppose that $(R, m)$ is a normal isolated singularity, and let $E_1, \ldots E_n$ be a collection of $R$-modules which are locally free on $spec\, R - \{m\}$. Let $F = \mathbb{S}^{\lambda_1}(E_1) \otimes \ldots \mathbb{S}^{\lambda_n}(E_n)$ where $\lambda_i \in Pos(dim E_i \otimes K)$. Let $f : X \to spec\, R$ is a resolution of singularities which frees the $E_i$, and let*

$$\tilde{F} = \mathbb{S}^{\lambda_1}(f^*E_1/torsion) \otimes \ldots \mathbb{S}^{\lambda_n}(f^*E_n/torsion).$$

*Then*

$$H^i_m(F^*) \cong H^{i-1}(X, \tilde{F}^*)$$

*for $2 \le i \le dim\, R - 1$. In particular,*

$$depth\, F^* = max\{i \mid H^j(X, \tilde{F}^*) = 0, \forall 1 \le j \le i-2\}$$

*provided the set on right is nonempty (otherwise $depth\, F^* = 2$).*

*Proof.* Consider the spectral sequence

$$E_2^{pq} = H^p_m(R^q f_*\tilde{F}^*) \Rightarrow H^{p+q}_D(\tilde{F}^*)$$

If we plot the $E_2$ terms in the $pq$-plane, then our assumptions imply that the nonzero terms are concentrated along the $p$ and $q$ axes. Since the abutment vanishes for $p + q < dim\, R$, we can deduce equalities

$$H^i_m(f_*\tilde{F}^*) = 0$$

for $i < 2$, and isomorphisms

$$H^{i-1}(X, \tilde{F}^*) \cong E_i^{0,i-1} \xrightarrow{\sim} E_i^{i,0} \cong H^i_m(f_*\tilde{F}^*)$$

for $i \geq 2$. The vanishing of the first two local cohomologies implies that $f_*\tilde{F}^*$ is reflexive. Since it coincides with the reflexive sheaf $F^*$ on the punctured spectrum, it coincides everywhere. □

We now return to the global setting. Let $X$ be a smooth projective variety.

**Lemma 4.3.** *Let $i : F \to E$ be a map of vector bundles (not necessarily of constant rank) on $X$ and let $L$ be another line bundle. Suppose either that $F$ is nef and $L$ is nef and big, or the other way around. If $\lambda \in Pos(rk(i(F) \otimes \mathbb{C}(X)))$, then there exists a geometrically acyclic sheaf $\mathcal{F}$ such that*

$$\omega_X \otimes image(\mathbb{S}^\lambda(F)) \otimes L \subseteq \mathcal{F} \subseteq \omega_X \otimes \mathbb{S}^\lambda(E) \otimes L.$$

*The first inclusion is an isomorphism over the open set where $i(F)$ is locally free.*

*Proof.* Choose a birational map $f : Y \to X$ which frees $i(F)$ and with $Y$ smooth. Then $\tilde{F} = f^*i(F)/torsion$ is nef or nef and big in accordance with our assumptions about $F$. Therefore $\mathcal{F}' = \omega_Y \otimes \mathbb{S}^\lambda(\tilde{F}) \otimes f^*L$, and hence also $\mathcal{F} = f_*\mathcal{F}'$, is geometrically acyclic. There is an injection $\mathcal{F}' \hookrightarrow \omega_Y \otimes \mathbb{S}^\lambda(f^*E) \otimes f^*L$ and a surjection $\omega_Y \otimes f^*\mathbb{S}^\lambda(F) \otimes f^*L \to \mathcal{F}'$ which yield the inclusions after applying $f_*$. □

**Corollary 4.4.** *Let $b$ be the dimension of the "base locus" of $E$ i.e. the dimension of the support of $coker[H^0(E) \otimes O_X \to E]$ (or 0 if the support is empty). If $L$ is a nef and big line bundle and $\lambda \in Pos(rk(E))$, then*

$$H^i(X, \omega_X \otimes \mathbb{S}^\lambda(E) \otimes L) = 0$$

*for $i > b$.*

## 5. Appendix: Schur-Weyl Functors

We will briefly describe the Schur-Weyl functors [FH, pp 231-7] which generalize the exterior and symmetric powers. A partition is a nonincreasing sequence of natural numbers $\lambda = (\lambda_1, \lambda_2, \dots)$ which is eventually zero. Its length $length(\lambda)$ is the largest $n$ with $\lambda_n \neq 0$, and $|\lambda| = \sum_i \lambda_i$. As usual a partition can be represented by a Young diagram, which consists of $|\lambda|$ boxes arranged so that $\lambda_i$ boxes lie in the $i$th row. For each diagram, we will need to choose a labeling of the boxes by integers from 1 to $|\lambda|$ so as to get a Young tableau; for our purposes it will not matter how this is done. Then every partition will determine an idempotent $e_\lambda \in \mathbb{Q}[S_{|\lambda|}]$ obtained by normalizing the so called Young symmetrizer. Given a finite dimensional complex vector space $E$, define the Schur-Weyl power by

$$\mathbb{S}^\lambda E = E^{\otimes|\lambda|} e_\lambda$$

where the symmetric group acts on the right by permuting factors.

The $\mathbb{S}^\lambda(E)$ are a complete set of representatives for the irreducible representations of $SL(E)$. According to Borel-Bott-Weil theory, these representations can be realized geometrically as spaces of sections of line bundles on the variety $Flag(E)$ of complete flags in $E$. Let $\pi_k : Flag(E) \to Grass_k(E)$ be the canonical map to the Grassmanian of $k$-dimensional *quotients* of $E$, and $i_k$ its Plücker embedding. Then

$$\mathbb{S}^\lambda(E) = H^0(Flag(E), \bigotimes_k \pi_k^* i_k^* O_{\mathbb{P}(\wedge^k E)}(a_k))$$

where $a_i = \lambda_i - \lambda_{i+1}$. This shows, among other things, that $\mathbb{S}^\lambda(E) = 0$ as soon as $length(\lambda) > dim(E)$. We can also replace $Flag(E)$, above, by the partial flag

variety $Flag_{k_1,\dots,k_m}(E)$ parameterizing flags of $E \supset E_{k_1} \supset E_{k_2} \dots$ where $E_{k_i}$ has codimension $k_i$ and $\{k_i\}$ is the set of the indices where $a_{k_i} \neq 0$. For a treatment along these lines see [F2, chap. 9].

All of this makes perfect sense when $E$ is a replaced with a vector bundle, provided we replace $Grass_k$ and $Flag$ by the corresponding bundles. The validity of all of the above assertions for $E$ follows by standard base change arguments.

We will need an extension of the Schur-Weyl functors to coherent sheaves where the above formulas are unsuitable. Define

$$\mathbb{S}^\lambda(E) = \frac{E^{\otimes|\lambda|}}{(1-e_\lambda)E^{\otimes|\lambda|}} \cong E^{\otimes|\lambda|} \otimes_{\mathbb{Q}[S_{|\lambda|}]} \frac{\mathbb{Q}[S_{|\lambda|}]}{(1-e_\lambda)\mathbb{Q}[S_{|\lambda|}]}.$$

The isomorphism class of the module $\mathbb{Q}[S_{|\lambda|}]/(1-e_\lambda)\mathbb{Q}[S_{|\lambda|}]$ is independent of the choice of tableau, so the same goes for $\mathbb{S}^\lambda(E)$. This definition coincides with the previous one for locally free sheaves, and takes epimorphisms to epimorphisms, so it can be computed with the help of a presentation.

It will be convenient to set

$$\mathbb{S}^\lambda_+ E = \mathbb{S}^\lambda E \otimes (detE)^{length(\lambda)} = \mathbb{S}^{\lambda + length(\lambda)\mu}(E)$$

where $\mu = (1, 1, \dots 1, 0 \dots)$ ($rk(E)$ ones). For each positive integer $e$, let $Pos(e)$ denote the set of partitions $\lambda$ of length at most $e$, such that $\lambda_e$ is greater than or equal to the length of $(\lambda_1 - \lambda_e, \dots, \lambda_{e-1} - \lambda_e, 0, \dots)$.

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